"Your Turn" Solutions

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Hansen 1.10

Let event H denote an individual earning a high wage and event C denote whether an individual has a college degree. We know that the probability of earning a high wage if one has a college degree is 0.53. We also know that 36% of the population has a college degree, while 31% of the population earns a high wage.

Find the probability that an individual has a college degree if they earn a high wage. Then find the probability that an individual has a college degree given they earn a low wage.

Solution:

First, we list what we know:

$$\begin{aligned} P(C) &= 0.36 \\ P(H) &= 0.31 \\ P(H|C) &= 0.53 \end{aligned}$$

We need to find $P(C|H)$ and $P(C|H^C)$. Using Bayes' Rule, we can first find $P(C|H)$:

$$\begin{aligned} P(C|H) &= \frac{P(H|C)P(C)}{P(H)} \\ &= \frac{0.53 \cdot 0.36}{0.31} \\ &= 0.62 \end{aligned}$$

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Similarly, we can find $P(C|H^C)$:

$$\begin{aligned} P(C|H^C) &= \frac{P(H^C|C)P(C)}{P(H^C)} \\ &= \frac{0.47 \cdot 0.36}{0.69} \\ &= 0.25 \end{aligned}$$

Full House

Suppose you draw 5 cards from a standard playing deck. Find the probability of drawing a full house.

Solution

We first have to pick 1 value out of the 13 values available. From the 4 possible cards for that value, we must pick 3. Then we need to pick another value from the 12 remaining values. Of the 4 cards with this second value, we need to pick 2. Mathematically:

$$\begin{aligned} P(\text{Full House}) &= \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \\ &\approx 0.14\% \end{aligned}$$

Moments of Linear Transformations

Let X be a random variable with finite variance and a, b be constants. Let $Y = a + bX$. then:

$$\mathbb{E}[Y] = a + b\mathbb{E}[X] \quad (1)$$

$$Var(Y) = b^2 Var(X) \quad (2)$$

Solution

<u>Claim:</u> $\mathbb{E}[Y] = a + b\mathbb{E}[X]$.

Proof. Starting from the expectation of Y:

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[a + bX] \\
&= \int_{-\infty}^{\infty} (a + bx) f_X(x) dx \\
&= \int_{\infty}^{\infty} a f_X(x) dx + \int_{-\infty}^{\infty} bx f_X(x) dx \\
&= a \int_{-\infty}^{\infty} f_X(x) dx + b \int_{-\infty}^{\infty} x f_X(x) dx \\
&= a + b\mathbb{E}[X]
\end{aligned}$$

This proves the claim. ■

<u>Claim:</u> $Var(Y) = b^2 Var(X)$.

Proof. Starting with the variance of Y:

$$\begin{aligned}
Var(Y) &= \mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right] \\
&= \mathbb{E}\left[(a + bX - \mathbb{E}[a + bX])^2\right] \\
&= \mathbb{E}\left[(a + bX - a - b\mathbb{E}[X])^2\right] \\
&= \mathbb{E}\left[(bX - b\mathbb{E}[X])^2\right] \\
&= \mathbb{E}\left[b^2 X - 2b^2 X\mathbb{E}[X] + b^2\mathbb{E}[X]^2\right] \\
&= b^2\mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\
&= b^2\mathbb{E}\left[(X - \mu_x)^2\right] \\
&= b^2 Var(X)
\end{aligned}$$

this proves the claim. ■